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NON-LINEARLY DYNAMIC MODELLING OF AN AXIALLY MOVING BEAM WITH A TIP MASS

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In this paper, the equations of motion for a deploying beam with a tip mass are derived by using Hamilton's principle. In the dynamic formulations, the beam is divided into two parts. One part of the beam is outside the rigid support and is free to vibrate, while the remaining part is inside the support and is restrained from vibrating. Four dynamic models: Timoshenko, Euler, simple-flexible and rigid-body beam theories, are used to describe the axially moving beam. An external force, parallel to the direction of the axially moving motion, is applied at the left-hand side of the flexible beam. It is found that the axially moving motion and flexible vibrations are non-linearly coupled in the system equations. Finally, the effects of several conditions on the rigid-body motion and the flexible are discussed.

1. INTRODUCTION

Dynamic problems of axially moving beams are major concerns for various applications such as textile industry, tapes, band saws, belts and chains, robot arms and flexible appendages of a spacecraft. However, these problems have been traditionally investigated on the basis of the assumption that the member is a rigid body, but in fact, when the amplitudes of vibration are greater than the allowable limit, some problems will arise. To obtain a more precise anticipation of the motions of the axially moving beams, a dynamic analysis of the elastic beams is necessary.

The dynamic analysis of an axial moving beam has been studied extensively in the past 20 years. Mote [1] and Tabarrok *et al*. [2] studied the dynamics of an axially moving beam. Tsuchiya [3] analyzed the attitudes of a spacecraft with a rotor during extension of flexible appendages. Kane *et al*. [4] proposed an algorithm which can be used to predict the behavior of a beam when its base undergoes the general three-dimensional motions. Using a recent approach, Yuh and Young [5] have derived a time-varying partial differential equation and boundary conditions for an axially moving beam with rotation. By using the finite element approach, Stylianou and Tabarrok [6, 7] solved an axially moving beam problem. The elements change in length and are functions of time. Lee [8] exploited

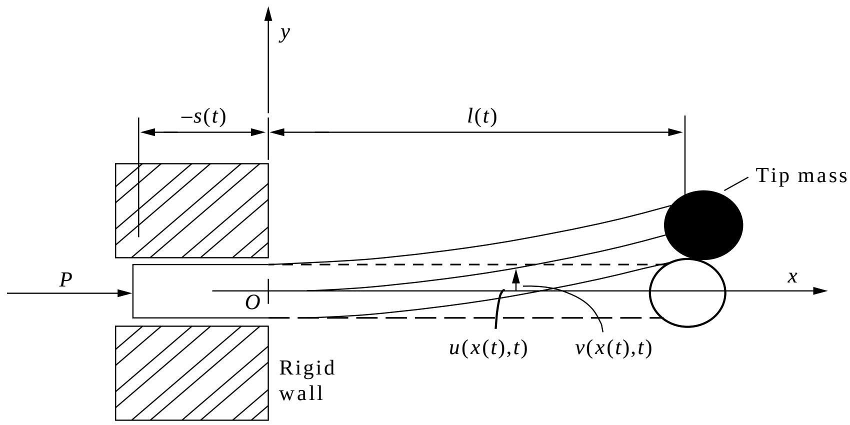

Figure 1. Schematic of a deploying beam system.

the properties of the eigenfunctions of a uniform fixed-free beam. The equations of motion in matrix form were formulated for the dynamic responses of an orthotropic rotating shaft which moved longitudinally over a spring support. In order to learn the deployment responses of a flexible beam, Creamer [9] presented a model using the Timoshenko beam theory in conjunction with base oscillatory motion. Tadikonda and Baruh [10] employed the Eurler beam theory to present a complete dynamic model for a translating flexible beam which deployed a payload from a fixed base. Al-Bedoor and Khulief [11] used the Euler beam theory to show the dynamic model for the vibrations of an elastic beam with prismatic and revolute joints. The model developed accounts for all the dynamic coupling terms, as well as the stiffening effect due to beam reference rotation.

In previous studies, most researchers employed the Euler beam theory to study the flexible vibrations [2, 5, 6, 11, 12, 14, 15] and the dynamic stability [1, 7, 13, 15] of an axially moving beam. However, the Timoshenko beam model, the tip mass effect, the external force and its corresponding axial motion were not formulated completely.

In this paper, the deployment of a flexible beam as shown in Figure 1 is considered. The flexible beam slides in and out of the rigid wall. At any instant, a part of the beam is outside the rigid support and is free to vibrate, while the remaining part of the beam is inside the wall and is restrained from the deformation in the transverse direction. Hamilton's principle [16] is employed to formulate the governing equations of the axial moving beam which is modelled by four separate beam models. In these formulations, an external force is applied, and the rigid-body motion and flexible vibrations are found to be non-linearly coupled.

2. FORMULATION OF THE GOVERNING EQUATIONS

There are four separate models, Timoshenko beam model, Euler beam model, simple-flexible model and rigid-body model, which can be used to describe an axially moving beam. Hamilton's principle is employed to derive the governing equations of motion for the system shown in Figure 1. The axially moving beam is supported by a rigid wall while a mass (M) is attached at its free end. An external force P is applied at the left-hand side of the beam. In what follows, the governing equations of the system are derived using Timoshenko beam theory which retains the effects of the shear deformation and rotary inertia. A reduction process of the system equations through the other three theories is sequentially presented.

2.1. TIMOSHENKO BEAM THEORY

The length of the beam outside the wall is $l(t)$ while the beam length inside the wall is $-s(t)$. The main point of the dynamic formulation is that the axially moving beam of the internal $x(t)\in\lfloor 0^+, l(t)\rfloor$ is free to vibrate while the other interval $x(t)\in\lfloor -s(t), 0^-\rfloor$ is constrained in the **j** direction. In Figure 1, a fixed Cartesian coordinate oxy is used to describe of the problem. It is assumed that the beam has mass density (ρ), flexural rigidity (EI) and cross-sectional area (A).

Position vector of any material point $(x(t), y)$ of the axially moving before deformation is

$$\mathbf{r} = x(t)\mathbf{i} + y\mathbf{j}, \tag{1}$$

where **i**, **j** are the unit vectors of the fixed coordinate. It is worth noting that the beam is deployable and $x(t)$ is a function of time.

The displacement field of the Timoshenko beam is

$$\mathbf{U} = [u(x(t), t) - y\psi(x(t), t)]\mathbf{i} + v(x(t), t)\mathbf{j}, \tag{2}$$

where $u(x(t), t)$ and $v(x(t), t)$ represent the axial and the transverse displacements of the beam respectively, and $\psi(x(t), t)$ is the slope of the deflection curve due to bending alone. It should be noted that the displacement $v(x(t), t)$ should be zero for $x(t)\in\lfloor -s(t), 0^-\rfloor$, since the rigid wall is the constraint of the axially moving beam in the **j** direction. Thus, (2) represents the displacement for the interval $\lfloor 0^+, \ell(t)\rfloor$.

The position vector of the point $(x(t), y)$ after deformation is

$$\mathbf{R}^f = \mathbf{r} + \mathbf{U}. \tag{3}$$

Taking total derivative of $\mathbf{R}^f(x(t), y, t)$ with respect to time, one obtains

$$\frac{D\mathbf{R}^f(x(t), y, t)}{Dt} = [\dot{x} + \dot{u} + \dot{x}u_x - y(\dot{\psi} + \dot{x}\psi_x)]\mathbf{i} + (\dot{v} + \dot{x}v_x)\mathbf{j}. \tag{4}$$

Therefore, the kinetic energies of the beam and the tip mass are, respectively,

$$K.E. = \tfrac{1}{2}\rho\int_v \frac{D\mathbf{R}^f}{Dt}\cdot\frac{D\mathbf{R}^f}{Dt}\,\mathrm{d}V = \int_{-s(t)}^{0^-} T_1\,\mathrm{d}x + \int_{0^+}^{l(t)} T_2\,\mathrm{d}x, \tag{5}$$

$$T_m = \tfrac{1}{2}M\frac{D\mathrm{R}^f}{Dt}\cdot\frac{D\mathrm{R}^f}{Dt}\bigg|_{(x(t),y)=(l(t),0)}$$

$$= \tfrac{1}{2}M[(\dot{x} + \dot{u} + \dot{x}u_x)^2 + (\dot{v} + \dot{x}v_x)^2]_{(x(t),y)=(l(t),0)}, \tag{6}$$

where

$$T_1 = \tfrac{1}{2}\{\rho A(\dot{x} + \dot{u} + \dot{x}u_x)^2 + \rho I(\dot{\psi} + \dot{x}\psi_x)^2\}, \tag{7}$$

$$T_2 = \tfrac{1}{2}\{\rho A[(\dot{x} + \dot{u} + \dot{x}u_x)^2 + (\dot{v} + \dot{x}v_x)^2] + \rho I(\dot{\psi} + \dot{x}\psi_x)^2\}. \tag{8}$$

The Lagrangian strains in the corresponding directions are

$$\varepsilon_{xx} = u_x - y\psi_x + \tfrac{1}{2}v_x^2, \qquad \varepsilon_{xy} = \tfrac{1}{2}(-\psi + v_x), \qquad \varepsilon_{yy} = 0, \tag{9}$$

where the higher order terms $\frac{1}{2}(u_x - y\psi_x)^2$ in ε_{xx}, $u_x\psi$ and $y\psi\psi_x$ in ε_{xy}, and $\frac{1}{2}\psi^2$ in ε_{yy} are neglected. The non-linear term $\frac{1}{2}v_x^2$ in (9) is due to the large deformation in the transverse direction. The total strain energy can be written as

$$S.E. = \tfrac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yy}\varepsilon_{yy})\,\mathrm{d}V = \int_{-s(t)}^{0^-} U_1^*\,\mathrm{d}x + \int_{0^+}^{l(t)} U_2^*\,\mathrm{d}x, \tag{10}$$

where

$$U_1^* = \tfrac{1}{2}[EAu_x^2 + EI\psi_x^2 + \tfrac{1}{4}EA\psi^2], \tag{11}$$

$$U_2^* = \tfrac{1}{2}[EA(u_x + \tfrac{1}{2}v_x^2)^2 + EI\psi_x^2 + \tfrac{1}{4}EA(v_x - \psi)^2], \tag{12}$$

and E is Young's modulus of the material.

In addition, the virtual work done by the external force P is defined as

$$\delta W = \mathbf{P}\cdot\delta\mathbf{R}^f|_{(-s(t),0)} = P[\delta(-s(t)) + \delta u(-s(t), t)]. \tag{13}$$

Finally, the variation of the kinetic energy (6) of the tip mass is

$$\int_{t_1}^{t_2} \delta T_m\,\mathrm{d}t = \int_{t_1}^{t_2} M\frac{D\mathbf{R}^f}{Dt}\cdot\delta\frac{D\mathbf{R}^f}{Dt}\bigg|_{(l(t),0)}\mathrm{d}t$$

$$= \left[M\frac{D\mathbf{R}^f}{Dt}\cdot\delta\mathbf{R}^f|_{(l(t),0)}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} M\frac{D^2\mathbf{R}^f}{Dt^2}\cdot\delta\mathbf{R}^f|_{(l(t),0)}\,\mathrm{d}t. \tag{14}$$

2.2. HAMILTON'S PRINCIPLE

The total Lagrangian function of the axially moving beam is

$$\mathscr{L}_b(t; \dot{x}, u_x, \dot{u}, v_x, \dot{v}, \psi, \psi_x, \dot{\psi}) = \int_{-s(t)}^{l(t)} (T_1 + T_2 - U_1^* - U_2^*)\,\mathrm{d}x$$

$$= \int_{-s(t)}^{0^-} \mathscr{L}_1\,\mathrm{d}x + \int_{0^+}^{l(t)} \mathscr{L}_2\,\mathrm{d}x, \tag{15}$$

where

$$\mathscr{L}_1 = \tfrac{1}{2}[\rho A(\dot{x} + \dot{u} + \dot{x}u_x)^2 + \rho I(\dot{\psi} + \dot{x}\psi_x)^2 - EA(u_x^2 + \tfrac{1}{4}\psi^2) - EI\psi_x^2], \tag{16}$$

$$\mathscr{L}_2 = \tfrac{1}{2}\{\rho A[(\dot{x} + \dot{u} + \dot{x}u_x)^2 + (\dot{v} + \dot{x}v_x)^2] + \rho I(\dot{\psi} + \dot{x}\psi_x)^2 - EA[(u_x + \tfrac{1}{2}v_x^2)^2 + \tfrac{1}{4}(v_x - \psi)^2] - EI\psi_x^2\}. \tag{17}$$

Notice that the displacement $v(x(t), t)$ in the **j** direction is zero for $x \in \lfloor -s(t), 0^- \rfloor$. Thus, $\mathscr{L}_1$ contains only the kinetic and strain energies in the **i** direction.

Consequently, the general form of Hamilton's principle for the system is

$$\int_{t_1}^{t_2} \left[\delta \int_{-s(t)}^{l(t)} \mathscr{L}_b \, \mathrm{d}x + \delta W + \delta T_m \right] \mathrm{d}t$$

$$= \int_{t_1}^{t_2} \left[\int_{-s(t)}^{0^-} \delta \mathscr{L}_1 \, \mathrm{d}x + \int_{0^+}^{l(t)} \delta \mathscr{L}_2 \, \mathrm{d}x + \mathscr{L}_1 \Big|_{-s(t)}^{0^-} \delta x + \mathscr{L}_2 \Big|_{0^+}^{l(t)} \delta x \right.$$

$$\left. + \delta W + \delta T_m \right] \mathrm{d}t = 0. \tag{18}$$

where $x(t)$ is also a dependent variable, because the equation of the axially moving motion will be derived.

By taking variation, applying the partical integration technique, using Leibnitz's rule and collecting the like terms, equation (18) can be rewritten as

$$0 = \int_{t_1}^{t_2} \left\{ \int_{-s(t)}^{0^-} \left(-\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_1}{\partial \dot{x}} \delta x \right) \mathrm{d}x + \int_{0^+}^{l(t)} \left(-\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{x}} \delta x \right) \mathrm{d}x \right.$$

$$+ \left[\left(\mathscr{L}_1 - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_1}{\partial \dot{x}} \right) \delta x \right]_{-s(t)}^{0^-} + \left[\left(\mathscr{L}_2 - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_2}{\partial \dot{x}} \right) \delta x \right]_{0^+}^{l(t)} + P\delta(-s(t))$$

$$+ \int_{-s(t)}^{0^-} \left[\left(-\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_1}{\partial \dot{u}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_1}{\partial u_x} \right) \delta u + \left(\frac{\partial \mathscr{L}_1}{\partial \psi} - \frac{\partial}{\partial t} \frac{\partial \mathscr{L}_1}{\partial \dot{\psi}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_1}{\partial \psi_x} \right) \delta \psi \right] \mathrm{d}x$$

$$+ \int_{0^+}^{l(t)} \left[\left(-\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{u}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_2}{\partial u_x} \right) \delta u + \left(-\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{v}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_2}{\partial v_x} \right) \delta v \right.$$

$$\left. \left. + \left(\frac{\partial \mathscr{L}_2}{\partial \psi} - \frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{\psi}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_2}{\partial \psi_x} \right) \delta \psi \right] \mathrm{d}x \right\} \mathrm{d}t$$

$$+ \int_{t_1}^{t_2} \left\{ \left[\left(\frac{\partial \mathscr{L}_1}{\partial u_x} - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_1}{\partial \dot{u}} \right) \delta u + \left(\frac{\partial \mathscr{L}_1}{\delta \psi_x} - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_1}{\partial \dot{\psi}} \right) \delta \psi \right]_{-s(t)}^{0^-} + P\delta u(-s(t), t) \right.$$

$$+ \left[\left(\frac{\partial \mathscr{L}_2}{\partial u_x} - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_2}{\partial \dot{u}} \right) \delta u + \left(\frac{\partial \mathscr{L}_2}{\partial v_x} - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_2}{\partial \dot{v}} \right) \delta v \right.$$

$$\left. \left. + \left(\frac{\partial \mathscr{L}_2}{\partial \psi_x} - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_2}{\partial \dot{\psi}} \right) \delta \psi \right]_{0^+}^{l(t)} - M \frac{D^2 \mathbf{R}^f}{Dt^2} \cdot \delta \mathbf{R}^f \big|_{(l(t),0)} \right\} \mathrm{d}t$$

$$+ \left\{ \int_{-s(t)}^{0^-} \frac{\partial \mathscr{L}_1}{\partial \dot{x}} \delta x \, \mathrm{d}x + \int_{0^+}^{l(t)} \frac{\partial \mathscr{L}_2}{\partial \dot{x}} \delta x \, \mathrm{d}x + \int_{-s(t)}^{0^-} \left(\frac{\partial \mathscr{L}_1}{\partial \dot{u}} \delta u + \frac{\partial \mathscr{L}_1}{\partial \dot{\psi}} \delta \psi \right) \mathrm{d}x \right.$$

$$\left. + \int_{0^+}^{l(t)} \left(\frac{\partial \mathscr{L}_2}{\partial \dot{u}} \delta u + \frac{\partial \mathscr{L}_2}{\partial \dot{v}} \delta v + \frac{\partial \mathscr{L}_2}{\partial \dot{\psi}} \delta \psi \right) \mathrm{d}x + M \frac{D\mathbf{R}^f}{Dt} \cdot \delta \mathbf{R}^f|_{(l(t),0)} \right\}_{t_1}^{t_2}, \tag{19}$$

where δx, δu, δv and $\delta \psi$ vanish at times t_1 and t_2 and δv equals zero at $x = 0^+$ and 0^-.

Since the axially moving beam is forced at the left-hand side, $x(t)$ is an unknown variable. Considering the axially moving motion, we have

$$\delta x = \delta(-s(t)) = \delta l(t). \tag{20}$$

Substituting equation (20) into equation (19), one obtains the axially moving motion

$$x: \quad \int_{-s(t)}^{0^-} \frac{\partial}{\partial t} \frac{\partial \mathscr{L}_1}{\partial \dot{x}} \mathrm{d}x + \int_{0^+}^{l(t)} \frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{x}} \mathrm{d}x - \left[\mathscr{L}_1 - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_1}{\partial \dot{x}} \right]_{-s(t)}^{0^-}$$

$$- \left[\mathscr{L}_2 - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_2}{\partial \dot{x}} \right]_{0^+}^{l(t)} + M(\ddot{x} + \ddot{u} + 2\dot{x}\dot{u}_x + \ddot{x}u_x + \dot{x}^2 u_{xx})|_{(l(t),0)} = P. \tag{21}$$

Subsequently, the governing equations of the flexible beam are

$-s(t) < x < 0^-$

$$u: \quad -\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_1}{\partial \dot{u}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_1}{\partial u_x} = 0, \tag{22}$$

$$\psi: \quad \frac{\partial \mathscr{L}_1}{\partial \psi} - \frac{\partial}{\partial t} \frac{\partial \mathscr{L}_1}{\partial \dot{\psi}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_1}{\partial \psi_x} = 0. \tag{23}$$

$0^+ < x < l(t)$

$$u: \quad -\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{u}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_2}{\partial u_x} = 0, \tag{24}$$

$$v: \quad -\frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{v}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_2}{\partial v_x} = 0, \tag{25}$$

$$\psi: \quad \frac{\partial \mathscr{L}_2}{\partial \psi} - \frac{\partial}{\partial t} \frac{\partial \mathscr{L}_2}{\partial \dot{\psi}} - \frac{\partial}{\partial x} \frac{\partial \mathscr{L}_2}{\partial \psi_x} = 0. \tag{26}$$

The associated boundary conditions are

$x = -s(t)$

$$-\left(\frac{\partial \mathscr{L}_1}{\partial u_x} - \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial \mathscr{L}_1}{\partial \dot{u}} \right)_{x=-s(t)} + P = 0, \tag{27a}$$

$$-\left(\frac{\partial\mathscr{L}_1}{\partial\psi_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_1}{\partial\dot\psi}\right)_{x=-s(t)}=0. \tag{27b}$$

$x=0$

$$\left[\frac{\partial\mathscr{L}_1}{\partial u_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_1}{\partial\dot u}\right]_{0^-}-\left[\frac{\partial\mathscr{L}_2}{\partial u_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_2}{\partial\dot u}\right]_{0^+}=0, \tag{28a}$$

$$\left[\frac{\partial\mathscr{L}_1}{\partial\psi_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_1}{\partial\dot\psi}\right]_{0^-}-\left[\frac{\partial\mathscr{L}_2}{\partial\psi_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_2}{\partial\dot\psi}\right]_{0^+}=0. \tag{28b}$$

$x=l(t)$

$$\left[\frac{\partial\mathscr{L}_2}{\partial u_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_2}{\partial\dot u}\right]_{x=l(t)}-M\left.\frac{D^2\mathbf{R}^f}{Dt^2}\right|_{(l(t),0,t)}=0, \tag{29a}$$

$$\left[\frac{\partial\mathscr{L}_2}{\partial v_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_2}{\partial\dot v}\right]_{x=l(t)}-M\left.\frac{D^2\mathbf{R}^f}{Dt^2}\right|_{(l(t),0,t)}=0, \tag{29b}$$

$$\left(\frac{\partial\mathscr{L}_2}{\partial\psi_x}-\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\mathscr{L}_2}{\partial\dot\psi}\right)_{x=l(t)}=0. \tag{29c}$$

Substituting the Lagrangian densities (16) and (17) into equations (21)–(29c), the governing equations become

$$\begin{aligned}
x:\quad &\int_{-s(t)}^{0^-}\{\rho A[(\ddot x+\ddot u+\ddot x u_x+\dot x\dot u_x)(1+u_x)+\dot u_x(\dot x+\dot u+\dot x u_x)]\\
&+\rho I(\ddot\psi\psi_x+\ddot x\psi_x^2+2\dot x\dot\psi_x\psi_x+\dot\psi\dot\psi_x)\}\,\mathrm{d}x\\
&+\int_{0^+}^{l(t)}\{\rho A[(\ddot x+\ddot u+\ddot x u_x+\dot x\dot u_x)(1+u_x)+\dot u_x(\dot x+\dot u+\dot x u_x)\\
&+(\ddot v v_x+\ddot x v_x^2+2\dot x\dot v_x v_x+\dot v\dot v_x)\rfloor+\rho I(\ddot\psi\psi_x+\ddot x\psi_x^2+2\dot x\dot\psi_x\psi_x+\dot\psi\dot\psi_x)\}\,\mathrm{d}x\\
&+M(\ddot x+\ddot u+2\dot x\dot u_x+\ddot x u_x+\dot x^2u_{xx})|_{(l(t),0,t)}\\
&+\tfrac12\{\rho A(2\dot x^2u_x+\dot x^2u_x^2+\dot x^2v_x^2-\dot u^2-\dot v^2)-\rho I\dot\psi^2\\
&+EA[(u_x^2+\tfrac12 v_x^2)^2+\tfrac14(v_x-\psi)^2]\}_{x=l(t)}-\tfrac12[\rho A\dot x^2u_x^2+\tfrac14 EA(v_x^2-2v_x\psi)]_{x=0^+}\\
&+[\tfrac12\rho A\dot x^2u_x^2]_{x=0^-}-\tfrac12\{\rho A(2\dot x^2u_x+\dot x^2u_x^2-\dot u^2)-\rho I\dot\psi^2\\
&+EA(u_x^2+\tfrac14\psi^2)\}_{x=-s(t)}=P.
\end{aligned} \tag{30}$$

$-s(t)<x<0^-$:

$$u:\quad -\rho A(\ddot x+\ddot u+\ddot x u_x+2\dot x\dot u_x+\dot x^2u_{xx})+EAu_{xx}=0, \tag{31}$$

$$\psi: \quad -\rho I(\ddot{\psi} + \ddot{x}\psi_x + 2\dot{x}\dot{\psi}_x + \dot{x}^2\psi_{xx}) - \tfrac{1}{4}EA\psi + EI\psi_{xx} = 0. \tag{32}$$

$0^+ < x < l(t)$:

$$u: \quad -\rho A(\ddot{x} + \ddot{u} + \ddot{x}u_x + 2\dot{x}\dot{u}_x + \dot{x}^2 u_{xx}) + EA(u_{xx} + v_x v_{xx}) = 0, \tag{33}$$

$$v: \quad -\rho A(\ddot{v} + \ddot{x}v_x + 2\dot{x}\dot{v}_x + \dot{x}^2 v_{xx}) + EA(u_{xx}v_x + u_x v_{xx} + \tfrac{3}{2}v_x^2 v_{xx}) + \tfrac{1}{4}EA(v_{xx} - \psi_x) = 0, \tag{34}$$

$$\psi: \quad -\rho I(\ddot{\psi} + \ddot{x}\psi_x + 2\dot{x}\dot{\psi}_x + \dot{x}^2\psi_{xx}) + \tfrac{1}{4}EA(v_x - \psi) + EI\psi_{xx} = 0. \tag{35}$$

The boundary conditions become

$x = -s(t)$:

$$u_x(-s(t), t) = -\frac{P}{EA}, \qquad \psi_x(-s(t), t) = 0. \tag{36a, b}$$

$x = 0$:

$$u(0^-, t) = u(0^+, t), \qquad v(0^+, t) = 0, \qquad \psi(0^-, t) = \psi(0^+, t),$$

$$u_x(0^-, t) = u_x(0^+, t) + \tfrac{1}{2}v_x^2(0^+, t), \qquad \psi_x(0^-, t) = \psi_x(0^+, t). \tag{37a, b, c, d, e}$$

$x = l(t)$:

$$0 = M(\ddot{x} + \ddot{u} + 2\dot{x}\dot{u}_x + \ddot{x}u_x + \dot{x}^2 u_{xx})|_{(l(t),0,t)} + [EA(u_x + \tfrac{1}{2}v_x^2)]_{x=l(t)}, \tag{38a}$$

$$0 = [-EA(u_x + \tfrac{1}{2}v_x^2)v_x - \tfrac{1}{4}EA(v_x - \psi)]_{x=l(t)} - M(\ddot{v} + 2\dot{x}\dot{v}_x + \ddot{x}v_x + \dot{x}^2 v_{xx})|_{(l(t),0,t)}, \tag{38b}$$

$$\psi_x(l(t), t) = 0. \tag{38c}$$

The non-linear partial differential equation (30)–(35) are the second-order derivatives of the variables x, u, v and ψ with respect to time. Equation (30) characterizes the axially moving motion of the beam while equations (31)–(35) describe the flexible vibrations of the deploying beam modeled by the Timoshenko beam theory. It is noticed that the axially moving motion and the flexible vibrations are non-linearly coupled. The terms $2\dot{x}\dot{u}_x$, $2\dot{x}\dot{v}_x$ and $2\dot{x}\dot{\psi}_x$ are the Coriolis forces in the u, v and ψ equations respectively.

2.3. EULER BEAM THEORY

If the Euler beam theory is used to describe the bending deformation of the axially moving beam by setting $\psi = v_x$ and neglecting the shear deformation and the rotating inertia effect of ρI term [17]. For the beam interval inside the wall, we have $v(x, t) = 0$, $x \in \lfloor -s(t), 0^- \rfloor$. Thus, the ψ term of the beam also vanishes. The governing equations of the system are

$$x: \quad \rho A \int_{s(t)}^{0^-} [(\ddot{x} + \ddot{u} + \ddot{x}u_x + \dot{x}\dot{u}_x)(1 + u_x) + \dot{u}_x(\dot{x} + \dot{u} + \dot{x}u_x)]\,\mathrm{d}x$$

$$+ \rho A \int_{0^+}^{l(t)} [(\ddot{x} + \ddot{u} + \ddot{x}u_x + \dot{x}\dot{u}_x)(1 + u_x) + \dot{u}_x(\dot{x} + \dot{u} + \dot{x}u_x)$$

$$+ (\ddot{v}v_x + \ddot{x}v_x^2 + 2\dot{x}\dot{v}_x v_x + \dot{v}\dot{v}_x)]\,\mathrm{d}x$$

$$+ M(\ddot{x} + \ddot{u} + 2\dot{x}\dot{u}_x + \ddot{x}u_x + \dot{x}^2 u_{xx})|_{(l(t),0,t)}$$

$$+ \tfrac{1}{2}[\rho A(2\dot{x}^2 u_x + \dot{x}^2 u_x^2 + \dot{x}^2 v_x^2 - \dot{u}^2 - \dot{v}^2) + EA(u_x + \tfrac{1}{2}v_x^2)^2]_{x = l(t)}$$

$$- \tfrac{1}{2}[EIv_{xx}^2]_{x = 0^+} - \tfrac{1}{2}[\rho A(2\dot{x}^2 u_x + \dot{x}^2 u_x^2 - \dot{u}^2) + EAu_x^2]_{x = -s(t)} = P. \tag{39}$$

$-s(t) < x < 0^-$:

$$u: \quad -\rho A(\ddot{x} + \ddot{u} + \ddot{x}u_x + 2\dot{x}\dot{u}_x + \dot{x}^2 u_{xx}) + EAu_{xx} = 0. \tag{40}$$

$0^+ < x < l(t)$:

$$u: \quad -\rho A(\ddot{x} + \ddot{u} + \ddot{x}u_x + 2\dot{x}\dot{u}_x + \dot{x}^2 u_{xx}) + EA(u_{xx} + v_x v_{xx}) = 0, \tag{41}$$

$$v: \quad -\rho A(\ddot{v} + \ddot{x}v_x + 2\dot{x}\dot{v}_x + \dot{x}^2 v_{xx}) + EA(u_{xx}v_x + u_x v_{xx} + \tfrac{3}{2}v_x^2 v_{xx}) - EIv_{xxxx} = 0. \tag{42}$$

The boundary conditions are

$x = -s(t)$:

$$u_x(-s(t), t) = -\frac{P}{EA}. \tag{43}$$

$x = 0$:

$$u(0^-, t) = u(0^+, t), \qquad u_x(0^-, t) = u_x(0^+, t), \qquad v(0^+, t) = v_x(0^+, t) = 0. \tag{44a, b, c, d}$$

$x = l(t)$:

$$[EA(u_x + \tfrac{1}{2}v_x^2)]_{x = l(t)} + M(\ddot{x} + \ddot{u} + 2\dot{x}\dot{u}_x + \ddot{x}u_x + \dot{x}^2 u_{xx})|_{(l(t),0,t)} = 0, \tag{45a}$$

$$[EA(u_x + \tfrac{1}{2}v_x^2)v_x - EIv_{xxx}]_{x = l(t)} + M(\ddot{v} + 2\dot{x}\dot{v}_x + \ddot{x}v_x)|_{(l(t),0,t)} = 0, \tag{45b}$$

$$v_{xx}(l(t), t) = 0. \tag{45c}$$

It is seen from the equations of motion (39)–(42) and the boundary conditions (43)–(45c) that the rigid-body motion and the flexural vibrations are non-linearly coupled.

Preveious studies of an axially moving beam have only focused attention on the transverse vibrations of the linear systems [1, 11, 12, 15]. As the rigid-body motion, axial deflection and nonlinear terms are neglected, the dynamic equations can be reduced to the same formulation.

2.4. SIMPLE-FLEXIBLE MODEL

In the simple-flexible model, one will eliminate the axial displacement $u(x, t)$ but retain the axially moving inertia effect. Equations (40), (41), (43) and (45a) contain the external force P, the tip mass M and the internal force of the beam. The reduction process is to carry these effects in the u equations into the v governing equation (42) and its boundary condition (46). Thus, one may define the internal axial forces as

$$p_1(x, t) = EAu_x, \qquad -s(t) \leqslant x \leqslant 0^- \tag{46}$$

$$p_2(x, t) = EA(u_x + \tfrac{1}{2}v_x^2), \qquad 0^+ \leqslant x \leqslant l(t). \tag{47}$$

The relationships of the external force at the left-hand side and the inertia force of the tip mass at the right-hand side are respectively,

$$p_1(-s(t), t) = EAu_x(-s(t), t) = -P, \tag{48}$$

$$p_2(l(t), t) = -M\ddot{x}. \tag{49}$$

Sequentially, equations (40)–(42) can be rewritten as

$$p_{1,x}(x, t) = \rho A\ddot{x}, \qquad -s(t) < x < 0^- \tag{50}$$

$$p_{2,x}(x, t) = \rho A\ddot{x}, \qquad 0^+ < x < l(t) \tag{51}$$

$$[p_2 v_x]_x - \rho A(\ddot{v} + \ddot{x}v_x + 2\dot{x}\dot{v}_x + \dot{x}^2 v_{xx}) - EIv_{xxxx} = 0, \qquad 0^+ < x < l(t). \tag{52}$$

As a result, we have

$$p_1(x, t) = p_1(-s(t), t) + \int_{-s(t)}^{x} \frac{\partial}{\partial x} p_1(x, t)\,\mathrm{d}x = -P + \rho A\ddot{x}(x + s(t)), \tag{53}$$

$$p_2(x, t) = p_2(l(t), t) - \int_{x}^{l(t)} \frac{\partial}{\partial x} p_2(x, t)\,\mathrm{d}x = -M\ddot{x} - \rho A\ddot{x}(l(t) - x). \tag{54}$$

From the physical property of the continuous axial force at $x = 0$, we have

$$p_1(0^-, t) = p_2(0^+, t), \tag{55}$$

and from equations (53) and (54), we obtain

$$\rho A(l(t) + s(t))\ddot{x} + M\ddot{x} = P. \tag{56}$$

Sybstituting equations (50)–(54) and (56), into equations (39) and (42), the governing equations are simplified as

$$x:\quad \rho A\int_{-s(t)}^{0^-}\ddot{x}\,\mathrm{d}x+\rho A\int_{0^+}^{l(t)}[\ddot{x}+(\ddot{v}v_x+\ddot{x}v_x^2+2\dot{x}\dot{v}_xv_x+\dot{v}\dot{v}_x)]\,\mathrm{d}x+M\ddot{x}$$

$$+\tfrac{1}{2}[\rho A(\dot{x}^2v_x^2-\dot{v}^2)-\tfrac{1}{2}M\ddot{x}v_x^2]_{x=l(t)}-\tfrac{1}{2}[EIv_{xx}^2]_{x=0^+}=P, \qquad (57)$$

$$v:\quad \rho A(\ddot{v}+2\dot{x}\dot{v}_x+\dot{x}^2v_{xx})+[M+\rho A(l(t)-x)]\ddot{x}v_{xx}+EIv_{xxxx}=0. \qquad (58)$$

The boundary conditions are

$x=0$:

$$v(0,t)=v_x(0,t)=0. \qquad (59\text{a, b})$$

$x=l(t)$:

$$\{M[\ddot{v}+2\dot{x}\dot{v}_x]-EIv_{xxx}\}_{x=l(t)}=0, \qquad (60\text{a})$$

$$v_{xx}(l(t),t)=0. \qquad (60\text{b})$$

2.5. RIGID-BODY MODEL

The governing equation of a rigid-body motion of the axially moving beam can be obtained by neglecting all the flexibility terms. Thus, we have

$$\rho A(l(t)+s(t))\ddot{x}+M\ddot{x}=P. \qquad (61)$$

3. DISCUSSION

The objective of this paper is to derive the equations of the axially moving beam system by use of various beam theories. The reduction process was shown by starting with the Timoshenko beam theory and going through the Euler beam theory, simple-flexible beam theory and, finally, the rigid-body beam.

Now, several important observations can be made:

(i) In the Timoshenko and Euler beam models, the external force P occurs in the nonhomogeneous term of the x governing equation (30, 39) and in the u boundary condition (36a, 43) at $x=-s(t)$, but P exists in the x governing equation (57) only for the simple-flexure and rigid-body models.

(ii) In the simple-flexible model, the v governing equation becomes linear. Boundary condition (60a) states the shear force balance and is complicated. The finite element method was employed by Stylianou and Tabarrok [7, 8] to investigate the dynamic response and stability of the axially moving beam with tip mass. However, many investigators [9–11, 15] employed the assumed modes method to solve the simple-flexible beam problem.

(iii) The axially moving motion and flexible vibration are non-linearly coupled in all the Timoshenko, Euler and simple-flexible beam models. Even though the external force is absent, the are also coupled. Thus, a complete analysis of the axially moving beam should include both the axially moving motion and the flexible vibration.

(iv) The external force P could be practically implemented by a hydraulic or motor driven system. In order to control the axially moving beam motion or suppress the system vibration, the effect of the external force needs to be understood in detail.

(v) Because of the coupling effect, the Coriolis forces exist in the governing equations. This phenomenon is also seen in a rotating disk [18] and traveling string [19].

4. CONCLUSIONS

In this paper, the axially moving motion, flexible vibration and boundary conditions for an axially moving beam with a tip mass are derived completely. The formulation is based on the expression of the kinetic and strain energies of the system by Hamilton's principle. In our formulation, the rigid wall prevents the vertical deflections but does not restrain the horizontal displacement. The velocity and acceleration of the deployment motion of the beam are also included in the formulae. It is found that the rigid-body motion and flexible vibration are non-linearly coupled. The Coriolis forces exist in this system.

The transient amplitude, steady-state response and dynamic stability analysis of the axially moving beam would be the interesting problems for further investigation.

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REFERENCES

1. C. D. MOTE, JR 1972 *Shock and Vibration Digest* **4**, 2–11. Dynamic stability of axially moving materials.
2. B. TABARROK, C. M. LEECH and Y. I. KIM 1974 *Journal of the Franklin Institute* **297**(3), 201–220. On the dynamics of axially moving beams.
3. K. TSUCHIYA 1983 *Journal of Guidance* **6**(2), 100–103. Dynamics of a spacecraft during extension of flexible appendages.
4. T. R. KANE, R. R. RYAN and A. K. BANERJEE 1987 *Journal of Guidance* **10**(2), 139–151. Dynamics of a cantilever beam attached to a moving base.
5. J. YUH and T. YOUNG 1991 *ASME Journal of Dynamic Systems, Measurement, and Control* **113**, 34–40. Dynamic modeling of an axially moving beam in rotation: simulation and experiment.
6. M. STYLIANOU and B. TABARROK 1994 *Journal of Sound and Vibration* **178**(4), 433–453. Finite element analysis of an axially moving beam, part I: time integration.
7. M. STYLIANOU and B. TABARROK 1994 *Journal of Sound and Vibration* **178**(4), 455–481. Finite element analysis of an axially moving beam, part II: stability analysis.
8. H. P. LEE 1995 *Journal of Sound and Vibration* **179**(2), 347–357. Flexural vibration of an orthotropic rotating shaft moving over supports.

9. N. G. Creamer 1990 *Journal of Guidance* **15**(2), 527–529. Deployment of a flexible beam from an oscillating base.
10. S. S. K. Tadikonda and H. Baruh 1992 *ASME Journal of Dynamic Systems, Measurement, and Control* **114**, 422–427. Dynamics and control of a translating flexible beam with a prismatic joint.
11. B. O. Al-Bedoor and Y. A. Khulief 1996 *Journal of Sound and Vibration* **190**(2), 195–206. Vibrational motion of an elastic beam with prismatic and revolute joints.
12. B. O. Al-Bedoor and Y. A. Khulief 1996 *Journal of Sound and Vibration* **192**(1), 159–171. An approximate analytical solution of beam vibrations during axial motion.
13. R. Elmaraghy and B. Tabarrok 1975 *Journal of the Franklin Institute* **300**(1), 25–39. On the dynamic stability of an axially oscillating beam.
14. P. K. C. Wang and J. D. Wei 1987 *Journal of Sound and Vibration* **116**(1), 149–160. Vibrations in a moving flexible robot arm.
15. R. J. Theodore, J. H. Arakeri and A. Ghosal 1996 *Journal of Sound and Vibration* **91**(3), 363–376. The modeling of axially translating flexible beams.
16. L. Meirovitch 1980 *Computational Methods in Structural Dynamics*. Rockville, MD: Sijthoff & Noordhoff.
17. S. Timoshenko, D. H. Young and W. Weaver Jr 1974 *Vibration Problem in Engineering*. New York: Wiley; 4th edition.
18. R. F. Fung 1997 *Journal of Sound and Vibration* **199**(2), 223–236. Vibration and variable structure control with integral compensation in non-constant rotating disk system.
19. R. F. Fung and S. L. Wu 1997 *Journal of Sound and Vibration* **204**(1), 171–179. Dynamic stability of a three-dimensional string subjected to both magnetic and tensioned excitations.